CHAPTER 4: IRREPS OF S_n and $\mathcal{U}_d$

§4.1 Minimal projections and irreducible representations

For a given (finite) group G, recall the group algebra $\mathbb{C}[G]$, the $\mathbb{C}$-vector space with basis $\{|g\rangle\}_{g\in G}$ and multiplication

$$\left(\sum_{g\in G} c_g |g\rangle\right)\left(\sum_{h\in G} d_h |h\rangle\right) = \sum_{g,h\in G} c_g d_h |gh\rangle .$$

$\boxed{\text{Def}}$ (Projections)

A projection in $\mathbb{C}[G]$ is an element $p\in\mathbb{C}[G]$ with $p^2=p$. A non-zero projection p is called minimal, if there are no non-zero projections q,r s.t. $p=q+r$. Two projections p and q are equivalent if there are invertible elements $x,y\in\mathbb{C}[G]$ s.t. $xpy=q$, and disjoint if $pzq=0$ for all $z\in\mathbb{C}[G]$.

$\boxed{\text{Def}}$ (Central projections)

A central projection in $\mathbb{C}[G]$ is a projection in

$Z(\mathbb{C}[G]) = \{x \in \mathbb{C}[G] : xy = yx \text{ for all } y \in \mathbb{C}[G]\}$.

A non-zero central projection is called minimal if it

cannot be written as a sum of non-zero central projections.

$\boxed{\text{Prop}}$ Let G be a finite group with group algebra $\mathcal{A} = \mathbb{C}[G]$.

Irreducible representations of G are in 1-1 correspondence with:

-) equivalence classes of minimal projections in $\mathcal{A}$.

-) minimal central projections in $\mathcal{A}$.

Let $(\varphi_\alpha, V_\alpha)$ be an irreducible representation of G with

character $\chi_\alpha(g) = \operatorname{tr} \varphi_\alpha(g)$. Then

$$p_\alpha = \frac{\dim V_\alpha}{|G|} \chi_\alpha$$

is the minimal central projection corresponding to $(\varphi_\alpha, V_\alpha)$.

<u>Proof idea:</u> Use the fact that $\mathbb{C}[G] \cong \bigoplus_\alpha \operatorname{End}(\mathbb{C}^{d_\alpha})$,

where α runs through the irreducible representations, and that

the center of $\operatorname{End}(\mathbb{C}^{d_\alpha})$ is 1-dimensional and spanned by χ_α.

"$\square$"

$\boxed{\text{Cor}}$ Let (φ, V) be a representation of a finite group G with isotypical decomposition $V \cong \bigoplus_\alpha V_\alpha$ and $V_\alpha = W_\alpha \oplus \dots \oplus W_\alpha$ for inequivalent irreducible representations W_α of G. Let χ_α be the character of W_α. Then

$$\pi_\alpha = \frac{\dim W_\alpha}{|G|} \sum_{g \in G} \overline{\chi_\alpha(g)}\, \varphi(g)$$

projects onto the isotypical component V_α of V.

§ 4.2 Conjugacy classes of the symmetric group

Recall from character theory (§2.2) that for a finite group G,

$$\#(\text{irreps of } G) = \#(\text{conjugacy classes of } G)$$

Conjugacy class: $g \sim h \;:\Leftrightarrow\; \exists s \in G$ s.t. $g = s h s^{-1}$.

This is an equivalence relation partitioning G into conjugacy classes $C_1, \dots, C_k$, i.e., $G = \bigsqcup_{i=1}^{k} C_i$

Facts about permutations:

1) Every permutation $\pi \in S_n$ can be written uniquely as a product of disjoint cycles, e.g., $\pi = (13)(2)(465) \in S_6$

The cycle type of a permutation $\pi \in S_n$ is the tuple of cycle lengths in non-increasing order.

Ex.: $\pi = (14)(236)(58)(7)$ has cycle type $(3,2,2,1)$.

2) Cycle types $(\lambda_1, \dots, \lambda_d)$ of a permutation $\pi \in S_n$ form an ordered partition of n: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d \geq 0$

$$\text{and } \sum_{i=1}^{d} \lambda_i = n.$$

We use the notation $\lambda \vdash_d n$ for an ordered partition of n into at most d parts.

Note: If $d < n$ then not all possible partitions / cycle types appear. This will be important later.

3) Two permutations π, π' are conjugate iff they have the same cycle type: Let $(i_1, \dots, i_k)$ be a cycle of length k and $\sigma \in S_n$ be arbitrary $(k \leq n)$, then

$$\sigma (i_1, \dots, i_k) \sigma^{-1} = (\sigma(i_1), \dots, \sigma(i_k))$$

4) It follows from 1) – 3) that the conjugacy classes of S_n (and hence its irreducible representations) are indexed by the ordered partitions of n into n parts.

§ 4.3 Constructing the irreducible representations of S_n and U_d

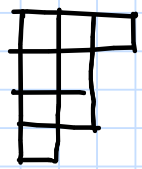
Irreducible representations of S_n $\leftrightarrow$ ordered partitions of n.

Def (Young diagrams and Young tableaux)

Let $\lambda \vdash_d n$ be a partition of n into at most d parts.

The Young diagram corresponding to $\lambda \vdash_d n$ is an arrangement of n boxes into d rows s.t. the i-th row has length λ_i.

Ex.: $\lambda = (3,2,2,1) \vdash_4 8$



A Young tableau is a Young diagram whose boxes are labeled with numbers $\{1, \dots, N\}$ where $N \neq n$ in general.

A standard Young tableau is a Young tableau with $N = n$ where the labels are increasing along rows (left to right) and along columns (top to bottom).

A semistandard Young tableau is a Young tableau whose labels are non-decreasing along rows and increasing along columns.

Ex.: Standard Young tableaux

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4\\ \cline{1-2}\end{array}$$

Semistandard Young tableaux with numbering $\{1,2\}$

$$\begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2&2\\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2&2\\ \cline{1-2}\end{array}$$

Schur-Weyl duality: $\left(\mathbb{C}^d\right)^{\otimes n} = \bigoplus_{\lambda \vdash_d n} V_\lambda \otimes U_\lambda$, where:

- the irrep V_λ of S_n has an ONB indexed by the set of standard Young tableaux of shape $\lambda \vdash_d n$.
- the irrep U_λ of $\mathcal{U}_d$ has an ONB indexed by the set of semistandard Young tableaux of shape $\lambda \vdash_d n$ and numbering $\{1, \dots, d\}$.

Recall: Every permutation $\pi \in S_n$ can be written as a product of at most $n-1$ transpositions (jk) with $1 \leq j < k \leq n$.

$\boxed{\textbf{Def}}$ Write $S_n \ni \pi = \tau_1 \cdot \ldots \cdot \tau_k$ for transpositions τ_i.
The sign of π is defined as $\operatorname{sgn}(\pi) = (-1)^k$.

Let now T be a standard Young tableau of shape $\lambda \vdash_d n$.

Define two subgroups $R_T, C_T \leq S_n$:

$$R_T := \{\pi \in S_n : \pi \text{ permutes integers within rows of } T\}$$

$$C_T := \{\pi \in S_n : \pi \text{ permutes integers within columns of } T\}.$$

Ex.:

T	$1\,2\,3$	$\begin{smallmatrix}1 & 2\\ 3 & \end{smallmatrix}$	$\begin{smallmatrix}1 & 3\\ 2 & \end{smallmatrix}$	$\begin{smallmatrix}1\\ 2\\ 3\end{smallmatrix}$
R_T	S_3	$\{e,(12)(3)\} \cong S_2$	$\{e,(13)(2)\} \cong S_2$	$\{e\} \cong S_1$
C_T	$\{e\} \cong S_1$	$\{e,(13)(2)\} \cong S_2$	$\{e,(12)(3)\} \cong S_2$	S_3

We define two elements in $\mathbb{C}[S_n]$:

$$r_T := \sum_{\pi \in R_T} \pi \qquad c_T := \sum_{\pi \in C_T} \operatorname{sgn}(\pi)\,\pi$$

$\boxed{\text{Def}}$ For given standard Young tableau T of shape $\lambda \vdash n$, the Young symmetrizer e_T is defined as $e_T := r_T c_T$.

Ex.: $\lambda = \square \ldots \square \vdash_d n$, then $C_T = \{e\}$, $R_T = S_n$, and $e_T = \sum_{\pi \in S_n} \pi$. For $\lambda = \begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}$, we have $e_T = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\pi$.

Prop Let T be a Young tableau of shape $\lambda \vdash n$, and let e_T be the corresponding Young symmetrizer. Then $f_T := \frac{d_\lambda}{n!} e_T$ is the minimal projection in $\mathbb{C}[S_n]$ corresponding to the irreducible representation V_λ of S_n, i.e., $V_\lambda \cong \mathbb{C}[S_n] e_T$. Here,

$$d_\lambda := \dim V_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$$

where for a box (i,j) in row i and column j of λ we define the hook length

$$h(i,j) = \#\text{boxes to the right of } (i,j) + \#\text{boxes below } (i,j) + 1.$$

The V_λ are called Specht modules. Every irrep of S_n is isomorphic to a Specht module V_λ for some $\lambda \vdash n$, and $V_\lambda \not\cong V_{\lambda'}$ for $\lambda \neq \lambda'$.

__Proof:__ See Christandl's PhD thesis or lecture notes by Alcock-Zeilinger.

Ex.: $\lambda =$

(1,2)

$h(1,2) = 3+2+1 = 6.$

$$d_\lambda = \frac{10!}{8\cdot 6\cdot 3\cdot 2\cdot 4\cdot 2\cdot 3} = 25\cdot 21 = 525$$

$\boxed{\text{Prop}}$ Let $|v\rangle \in V^{\otimes n}$, and for a standard Young tableau T of shape $\lambda \vdash n$ consider the Young symmetrizer e_T.

Let p be the number of parts of the partition λ (or the number of non-zero rows of the Young diagram λ).

i) If $p \leq d = \dim V$, then $\mathbb{C}[S_n]\, e_T |v\rangle$ is an irreducible representation of S_n isomorphic to the Specht module V_λ.

ii) If $p \leq d$, then $e_T V^{\otimes n}$ is an irreducible representation of $GL(V)$ (or U_d) on $V^{\otimes n}$. These are inequivalent for Young tableaux of different shape.

iii) Using the above, we have the Schur-Weyl decomposition of $V^{\otimes n}$ with $d = \dim V$ as an $S_n \times U_d$ representation:

$$V^{\otimes n} = \bigoplus_{\lambda \vdash_d n} V_\lambda \otimes U_\lambda .$$

Proof: Christandl.

The dimensions of V_λ and U_λ are given by:

$$d_\lambda = \dim V_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$$

$$m_\lambda = \dim U_\lambda = \prod_{1\le i<j\le d} \frac{\lambda_i - \lambda_j + j - i}{j-i}$$

The Schur-Weyl decomposition shows that

$$d^n = \sum_{\lambda \vdash_d n} d_\lambda m_\lambda .$$

Ex.: $\lambda =$  $= (5,2,2,1)$

$$d_\lambda = 525$$

$$m_\lambda = \frac{3+1}{1} \cdot \frac{3+2}{2} \cdot \frac{4+3}{3} \cdot \frac{0+1}{1} \cdot \frac{1+2}{2} \cdot \frac{1+1}{1}$$

$$= \overset{2}{\cancel{4}} \cdot \frac{5}{\cancel{2}} \cdot \frac{7}{\cancel{3}} \cdot 1 \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{2}{1}$$

$$= 70.$$